CHAPTER I

THE NUMERICAL RANGES

1.1 INTRODUCTION

This first chapter is largely expository. In it we develop the basic theory of numerical range. Many of the results are now well known, however, because of their importance and the frequent use we make of them, they are treated in some detail.

In section 2 we give F.F. Bonsall's definition of the numerical range of an element of a normed algebra, and, in the case of the algebra consisting of all the operators on some normed linear space, we see that two other numerical ranges can be defined, one is due to F.L. Bauer, the other to G. Lumer. We investigate the relationship between these numerical ranges and determine some of their properties. B. Bollobás [1] uses an extension of a result of Bishop and Phelps [1] to relate the numerical range of an operator, on a Banach space, to that of the closely connected conjugate operator. We show that when the space has a smooth dual the result follows quickly from the Bishop and Phelps' result.

The third section examines bounds on the numerical ranges. These bounds were first determined by Lumer. We use them to more closely relate the three numerical ranges defined in section 2.

From this we see that the numerical radius is the same for any of these three types of numerical range. We then use an argument of Bohnenblust and Karlin [1] to show the numerical radius is an equivalent linear norm for the algebra.

Bauer studied numerical ranges as a way of approximating the spectrum. We discuss the relationship between the numerical ranges and the spectrum in section 4, showing that the spectrum is contained in the closure of the numerical range. To approximate the spectrum more closely, we consider not just the numerical range defined in the algebra with the original norm, but numerical ranges defined relative to the algebra under equivalent norms. We show that the convex hull of the spectrum is the intersection of all such numerical ranges. This result was suggested by work of J.P. Williams [2] for operators on a Hilbert space, and a formula for the spectral radius given by Bonsall [2] (and in the case of a commutative algebra, by Bohnenblust and Karlin [1]). The result has also appeared in Bonsall and Duncan [1]. In the case when the algebra is the set of operators over a Hilbert space we sharpen the result to obtain an alternative proof of William's result.

In section 5 we restrict our attention to those elements with real numerical ranges. We give two characterizations of such elements, originally given by Lumer [2], and prove the fundamental result of I. Vidav [1] that for such elements the spectral and numerical radii are the same. We use this to obtain another characterization of elements with real numerical range, which will later form the basis of our theory in Chapter 2.

In the last section we use the set of elements with real numerical range to define several classes of elements of a normed algebra. Some properties of these various types of elements are then established, paving the way for our characterization of B^*-algebras in Chapter 2.

1.2 NUMERICAL RANGES

F.L. Bauer [1] in 1962 defined the numerical range of an operator over a finite dimensional normed linear space. His definition readily extends to any normed linear space E and may be approached as follows:

For any $x \in E$ let

$$D(x) = \{f \in E' : f(x) = \|f\|^2 = \|x\|^2\}$$

where E' denotes the dual space of E. The Hahn-Banach theorem ensures that $D(x) \neq \phi$. For any operator $T \in B(E)$ and $x \in E$ denote by $W_x(T)$ the convex subset of the complex plane defined by

$$W_x(T) = \{f(Tx) : f \in D(x)\}.$$

Then $W(T) = \bigcup\{W_x(T) : x \in E, \|x\| = 1\}$ is Bauer's numerical range. We will call it the spatial numerical range of T, which may also be defined as follows.

.2.1. **DEFINITION.** For a normed linear space E and $T \in B(E)$ the *spatial numerical range* of T is

$$W(T) = \{f(Tx) : x \in E, \|x\| = 1, \text{ and } f \in D(x)\}.$$

This definition was extended, from operators over a normed linear space, to arbitrary elements of a normed algebra A by F.F. Bonsall [1] who defined the numerical range of $a \in A$ as

$$V(a) = W(T_a)$$

where $a \mapsto T_a$ is the left regular representation of A in B(A), that is $T_a(b) = ab$ for all $b \in A$. We will call V(a) the algebra numerical range of a. A clearer definition of $V(a)$ is the following.

.2.2. **DEFINITION.** For a normed algebra A and $a \in A$, the *algebra numerical range* of a is

$$V(a) = \{f(ab) : b \in A, \|b\| = 1 \text{ and } f \in D(b)\}.$$

If E is a normed linear space, we may take $A = B(E)$ and define $V(T)$ for any $T \in A$.

.2.3. **PROPOSITION.** *For a normed linear space* E *and* $T \in B(E)$ *we have* $W(T) \subsetneqq V(T)$.

Proof. If $\lambda \in W(T)$ then $\lambda = f(Tx)$ for some $x \in E$, $\|x\| = 1$, and $f \in D(x)$. Let $U(y) = f(y)x$ for all $y \in E$ then $U \in B(E)$, and denote by g the evaluation functional defined by $g(S) = f(Sx)$ for all $S \in B(E)$, then $g \in D(U)$ and

$$\lambda = g(TU) \in V(T). \qquad //$$

Given a normed linear space E, for each $x \in E$ select an element $f_x \in D(x)$, and define mapping $\phi : x \mapsto f_x : E \to E'$. We call ϕ a support mapping of E into E'. We will say E is smooth if there is only one possible support mapping, that is if $D(x)$ is singleton for every $x \in E$, and rotund if every possible support mapping is one to one, or $D(x) \cap D(y) = \phi$ whenever $x \neq y$. (This is equivalent to the more usual definition of rotundity, viz. $x \neq y$ and $\|x\|, \|y\| \leqslant 1$ implies $\|x + y\| < 2$ [Giles, 1]).

A different, but related, definition of numerical range for an operator, over a normed linear space, was given by G. Lumer [1] in 1961 as follows.

.2.4. **DEFINITION.** For a normed linear space E, support mapping $\phi : x \mapsto f_x$ of E into E' and T ε B(E) define $\underline{W_\phi(T)}$ by

$$W_\phi(T) = \{f_x(Tx) : x \in E, \|x\| = 1\}.$$

Clearly, for any support mapping ϕ of E into E', $W_\phi(T) \subseteq W(T)$, in fact $W(T) = \bigcup_\phi W_\phi(T)$. If E is a smooth space, then $W_\phi(T) = W(T)$ where ϕ is the unique support mapping of E into E'.

We also see that if the space is a Hilbert space then $W(T) = W_\phi(T)$ is the traditional numerical range, first defined by Toeplitz in 1918 for an operator T over a finite dimensional inner-product space. Thus if H is a Hilbert space and T ε B(H) then

$$W(T) = \{(Tx,x) : x \in H, \|x\| = 1\} \qquad .....(1)$$

that is, W(T) is the image of the unit sphere of H under the quadratic form associated with T. In this setting W(T) is sometimes referred to as the "Toeplitz field of values" of T.

We say that a normed algebra A has a *one* e ε A, if ea = ae = a for all a ε A, further if $\|e\| = 1$ we say A is a *unital Banach Algebra* with *unit*, e.

It is well known [Halmos,1] that the numerical range of an operator over a Hilbert space, as defined by (1), is convex. In general, this is not true for either $W_\phi(T)$ or W(T) when T is an operator over an arbitrary normed linear space, see [Lumer, 1] and [Bonsall and Duncan, 1, 11.1]. However as the following well known result [Bonsall, 1] will show, $V(a)$ is convex if A is a unital normed algebra. In fact, if the algebra has a unit, a much richer theory of numerical range can be developed. It is for this reason that we will largely restrict our attention to unital algebras.

.2.5. **LEMMA.** *For a unital normed algebra* A *and* $a \in A$

$$V(a) = \{f(a) : f \in D(e)\}$$

Proof. Clearly $\{f(a) : f \in D(e)\} \subseteq V(a)$.

Conversely if $\lambda \in V(a)$ then $\lambda = f(ab)$ for some $b \in A$, $\|b\| = 1$, and $f \in D(b)$. Now $\omega_b \in A'$ defined by $\omega_b(u) = f(ub)$, for all $u \in A$, is an element of $D(e)$ and $\lambda = \omega_b(a)$ so $\lambda \in \{f(a) : f \in D(e)\}$. //

.2.5.1. **COROLLARY.** *For a unital normed Algebra* A *and* $a \in A$, $V(a)$ *is a closed convex subset of* C.

Proof. If $\lambda_1, \lambda_2 \in V(a)$ then $\lambda_i = f_i(a)$ for $f_i \in D(e)$, $i = 1,2$, so for $0 \leqslant \alpha \leqslant 1$, $\alpha\lambda_1 + (1-\alpha)\lambda_2 = (\alpha f_1 + (1-\alpha)f_2)(a)$ but $\alpha f_1 + (1-\alpha)f_2 \in D(e)$ and therefore $V(a)$ is convex.

$V(a)$ is the image of the weak*-compact subset $D(e)$ of A' under the weak*-continuous map $f \mapsto f(a) : A' \to C$, and so is compact, and therefore closed. //

.2.5.2. **COROLLARY.** *If* B *is a subalgebra of a unital normed algebra* A, *and if the unit* $e \in B$, *then for any* $a \in B$, $V_B(a) = V_A(a)$ *where the subscript denotes the algebra relative to which the algebra numerical range has been defined.*

Proof. If $f \in D_A(e)$ then $f|_B \in D_B(e)$ so $V_A(a) \subseteq V_B(a)$. Conversely if $f \in D_B(e)$ then, by the Hahn-Banach theorem, f can be extended to an element of $D_A(e)$. Therefore $V_B(a) \subseteq V_A(a)$ and so they are equal.//

The last corollary shows that if a is an element of a unital normed algebra then V(a) depends only on the subalgebra generated by a and e. It also shows that if A is not complete then V(a) is unaltered by passing to the completion of A. Thus in most cases we may, without loss of generality, assume A to be a Banach algebra.

From their definitions it is easily seen that the numerical ranges have the subadditive properties

$$W(\lambda T) = \lambda W(T) \text{ for all } \lambda \in C$$

and

$$W(T+V) \subseteq W(T) + W(V) \text{ for any } T, V \in B(E),$$

and similarly for $W_\phi(T)$ and $V(T)$, where $\lambda W(T)$ and $W(T) + W(V)$ have the obvious meanings

$$\lambda W(T) = \{\lambda\xi : \xi \in W(T)\}$$

and

$$W(T) + W(V) = \{\lambda+\xi : \lambda \in W(T) \text{ and } \xi \in W(V)\}.$$

Bishop and Phelps [1] have proved that for a Banach space X, $\bigcup\{D(x) : x \in X\}$ is dense in X', the property is called *subreflexivity*.

Bollobás [1] has shown that the proof of subreflexivity, given by Bishop and Phelps, in fact, leads to an extension of their result. Using his extended form of their result Bollobas proves that for any $T \in B(X)$ $\overline{W(T)} = \overline{W(T')}$, where T' denotes the conjugate operator of T defined by $T'f(x) = f(Tx)$ for all $x \in X$ and $f \in X'$.

We proe that if X is assumed to have a smooth dual the connection between $W(T)$ and $W(T')$ follows readily from the subreflexivity of X.

.2.6. **LEMMA.** *For a Banach Space* X *with a smooth dual, and for any* $T \in B(X)$ *we have*

$$\overline{W(T')} = \overline{W(T)}$$

Proof. If $\lambda \in W(T)$ then $\lambda = f(Tx)$ for some $x \in X$, $\|x\| = 1$, and $f \in D(x)$. Therefore

$$\begin{aligned}\lambda &= T'f(x)\\ &= \hat{x}(T'f) \in W(T')\end{aligned}$$

so $W(T) \subseteq W(T')$. It is thus sufficient to show $W(T') \subseteq \overline{W(T)}$. Now for $\lambda \in W(T')$ there exists $f \in X'$, $\|f\| = 1$, and $F \in D(f)$ such that $\lambda = F(T'f)$. By the subreflexivity of X there exists a sequence f_n in X' and x_n in X, $\|x_n\| = 1$ for all n, such that $\|f_n - f\| \to 0$ and $f_n \in D(x_n)$. But the sequence $\hat{x}_n$ has an X'-weak convergent subsequence, $\hat{x}_m$, that is $\hat{x}_m(g) \to G(g)$ for all $g \in X'$ and some

$G \in X''$, $\|G\| \leq 1$.

However $|\hat{x}_m(f) - 1| \leq |\hat{x}_m(f) - \hat{x}_m(f_m)| + |\hat{x}_m(f_m) - 1|$ so $\hat{x}_m(f) \to 1$ and $G \in D(f)$, but X' is smooth and so $G = F$.

Now

$$| \hat{x}_m(T'f_m) - F(T'f)| \leq |\hat{x}_m(T'f_m) - \hat{x}_m(T'f)| + |\hat{x}_m(T'f) - F(T'f)|$$

and so $\quad \hat{x}_m(T'f_m) \to F(T'f)$

or $\quad f_m(Tx_m) \to \lambda \quad$ and so $\lambda \in \overline{W(T)}$. //

In this section we have defined the numerical range of an element of a normed algebra and proved some of its properties, particularly for unital algebras. When the algebra is the set of operators over a normed linear space we have seen how three different types of numerical range can be defined for any operator. Much of our later theory will be developed in terms of the algebra numerical range, the spatial numerical range only being considered when the results are explicitly relevant to it, while $W_\phi(T)$ will not be considered after the next section. The reason for this will become apparent after the relationship between $W_\phi(T)$, $W(T)$ and $V(T)$ is further clarified in the next section.

1.3 BOUNDS ON THE NUMERICAL RANGES.

We now obtain several bounds on the numerical ranges as subsets of the complex plane.

We begin with a lemma first proved by Lumer [1, Lemma 12]. It provides the starting point for much of our later theory as well as establishing the relationship between $W_\phi(T)$, $W(T)$ and $V(T)$. The proof we give is an adaption of that given by Bonsall and Duncan [1, 2.5] for the corresponding result on $V(a)$.

.3.1. **LEMMA.** *For a normed linear space* E, *support mapping* ϕ *of* E *into* E′, *and* $T \in B(E)$ *we have that*

$$\sup \operatorname{Re} W_\phi(T) = \lim_{\alpha \to 0+} \alpha^{-1}\{\,\|I + \alpha T\| - 1\},$$ *the upper Gâteaux differential of the norm at* I *in the direction* T.

Proof. For any $x \in E$ let $\underline{x} = x/\|x\|$, then for $0 < \alpha < \|T\|^{-1}$ and $f = \phi(\underline{x})$ we have

$$\begin{aligned} \|[I-\alpha T]\underline{x}\| &\geq |f([I-\alpha T]\underline{x})| \quad \text{since } \|f\| = 1 \\ &\geq \operatorname{Re} f([I-\alpha T]\underline{x}) \\ &= 1 - \alpha \operatorname{Re} f(T\underline{x}) \quad \text{since } \alpha \in \mathbf{R} \text{ and } f \in D(\underline{x}) \\ &> 0 \quad \text{since } \alpha < \|T\|^{-1} \end{aligned}$$

Therefore $$\frac{\|[I-\alpha T]\underline{x}\|}{1-\alpha \operatorname{Re} f(T\underline{x})} \geq 1$$

and so

$$\frac{\|[I-\alpha T]x\|}{1-\alpha \operatorname{Re} f(T\underline{x})} \geq \|x\| \quad \text{for any } x \in E.$$

Now choose $x = [I + \alpha T]y$ where $y \in E$, $\|y\| = 1$ and we obtain

$$\frac{\|y-\alpha^2T^2y\|}{1 - \alpha \operatorname{Re} f(T\underline{x})} = \frac{1 + O(\alpha^2)}{1 - \alpha \operatorname{Re} f(T\underline{x})} \geq \|[I + \alpha T]y\|.$$

Therefore $\dfrac{\alpha \text{Re } f(T\underline{x}) + 0(\alpha^2)}{1 - \alpha \text{ Re } f(T\underline{x})} \geqslant \|[I + \alpha T]y\| - 1$

and since $\alpha > 0$

$$\frac{\text{Re } f(T\underline{x}) + 0(\alpha)}{1 - \alpha \text{ Re } f(T\underline{x})} \geqslant \alpha^{-1} \{ \|[I + \alpha T]y\| - 1\}.$$

Since the limit as α tends to 0 from above is easily seen to exist, we have

$$\text{Re } f(T\underline{x}) \geqslant \lim_{\alpha\to 0+} \alpha^{-1} \{ \|[I + \alpha T]y\| - 1\}, \text{ where}$$

$y \in E$, $\|y\| = 1$ and $x = [I + \alpha T]y$

but $\quad \text{Re } f(T\underline{x}) \in \text{Re } W_\phi(T)$ so

$$\text{Sup Re } W_\phi(T) \geqslant \lim_{\alpha\to 0+} \alpha^{-1} \{ \|[I + \alpha T]y\| - 1\} \text{ for all}$$

$y \in E$, $\|y\| = 1$, and since y may be chosen so that $\|[I + \alpha T]y\|$ is arbitrarily close to $\|I + \alpha T\|$ we have that

$$\text{Sup Re}_\phi(T) \geqslant \lim_{\alpha\to 0+} \alpha^{-1}\{ \|I + \alpha T\| - 1\}.$$

Conversely for any $x \in E$, $\|x\| = 1$, $f = \phi(x)$ we have

$1 + \alpha\text{Re } f(Tx) = \text{Re } f([1 + \alpha T]x) \leqslant \|I + \alpha T\|$ for all $\alpha > 0$.

Therefore

$$\text{Re } f(Tx) \leqslant \alpha^{-1}\{ \|I + \alpha T\| - 1\} \text{ for all } \alpha > 0$$

and so

$$\text{Sup Re } W_\phi(T) \leqslant \alpha^{-1} \{ \|I + \alpha T\| - 1\} \text{ for all } \alpha > 0$$

in particular

$$\text{Sup Re } W_\phi(T) \leqslant \lim_{\alpha\to 0} \alpha^{-1} \{ \|I + \alpha T\| - 1\},$$

proving the result. //

.3.1.1. **COROLLARY.** $\operatorname{Sup} \operatorname{Re} W(T) = \lim_{\alpha \to 0+} \alpha^{-1} \{ \|I + \alpha T\| - 1\}$

Proof. $\operatorname{Sup} \operatorname{Re} W(T) = \operatorname{Sup}_{\phi} \operatorname{Sup} \operatorname{Re} W_{\phi}(T) = \lim_{\alpha \to 0+} \alpha^{-1} \{ \|I + \alpha T\| - 1\}$. //

.3.1.2. **COROLLARY.** *For a unital Banach Algebra* A *and* $a \in A$

$$\operatorname{Sup} \operatorname{Re} V(a) = \lim_{\alpha \to 0+} \alpha^{-1} \{ \|e + \alpha a\| - 1\}.$$

Proof. Let $a \to T_a$ be the left regular representation of A in B(A). Then $V(a) = W(T_a)$ and so from Lemma .3.1.

$$\operatorname{Sup} \operatorname{Re} (V(a) = \lim_{\alpha \to 0+} \alpha^{-1} \{ \|I + \alpha T_a\| - 1\}$$

$$= \lim_{\alpha \to 0+} \alpha^{-1} \{ \|e + \alpha a\| - 1\}, \text{ since A is a}$$

unital algebra and so $a \mapsto T_a$ is isometric. //

.3.1.3. **COROLLARY.** *For a normed linear space* E*, support mapping* ϕ *of* E *into* E'*, and* $T \in B(E)$ *we have* $V(T) = \overline{co}W(T) = \overline{co}W_{\phi}(T)$, *where* $\overline{co}$ *denotes closed convex hull.*

Proof. By Corollary .2.5.1. $\overline{co}W_{\phi}(T) \subseteq \overline{co}W(T) \subseteq V(T)$ since $W_{\phi}(T) \subseteq W(T) \subseteq V(T)$, so it is enough to prove $V(T) \subseteq \overline{co}W_{\phi}(T)$. Now since $\overline{co}W_{\phi}(T)$ is the intersection of all the closed half-planes containing $W_{\phi}(T)$ and since $W_{\phi}(\alpha T + \beta I) = \alpha W_{\phi}(T) + \beta$ and $V(\alpha T + \beta I) = \alpha V(T) + \beta$ for all α, $\beta \in C$, it is sufficient to prove that if $W_{\phi}(T)$ is contained in the left-half plane then so is $V(T)$. That is $\sup \operatorname{Re} V(T) \leqslant 0$ whenever $\operatorname{Sup} \operatorname{Re} W_{\phi}(T) \leqslant 0$, but this follows from Lemma .3.1. and Corollary .3.1.2. //

The next lemma, first proved by Bohnenblust and Karlin [1], enables us to equivalently reformulate Lemma .3.1. in an important way. For an element of a unital Banach algebra we define exp(a) by $\exp(a) = e + \sum_{n=1}^{\infty} a^n/n!$ [Rickart, 1]. This map of A into A has many of the properties of the real function with the same coefficients in its Taylor expansion; for example, if a, b $\in$ A commute then

$\exp(a + b) = \exp(a)\exp(b)$.

.3.2. LEMMA. *For a unital Banach Algebra* A *and* $a \in A$ *we have* $\lim_{\alpha\to 0+} \alpha^{-1}\{\|e + \alpha a\| - 1\} = \lim_{\alpha\to 0+} \alpha^{-1}\log\|\exp(\alpha a)\|$

Proof. Since $\sup \operatorname{Re} V(a) \leq \|a\|$ we see by Lemma .3.1. that $\{\|e + \alpha a\| - 1\} = 0(\alpha)$ for small $\alpha > 0$.

Also $\exp(\alpha a) = e + \alpha a + R_1$ where

$$R_1 = \sum_{n=2}^{\infty} (\alpha a)^n/n! \text{ and } \|R_1\| = 0(\alpha^2).$$

Therefore $\|e + \alpha a\| - \|R_1\| \leq \|\exp(\alpha a)\| \leq \|e + \alpha a\| + \|R_1\|$ for small α.

Now $\log(t)$ is a monotonically increasing function of $t > 0$ therefore $\log(\|e + \alpha a\| - \|R_1\|) \leq \log\|\exp(\alpha a)\| \leq \log(\|e + \alpha a\| + \|R_1\|)$.

Further $\log t = (t - 1) - (t - 1)^2/2 + \ldots$ for $t > 0$ so

$\log(\|e + \alpha a\| - \|R_1\|) = \|e + \alpha a\| - 1 - \|R_1\| + R_2$ where

$R_2 = \sum_{n=2}^{\infty} (-)^{n+1}(\|e + \alpha a\| - 1 - \|R_1\|)^n/n = 0(\alpha^2)$ similarly

$\log(\|e + \alpha a\| + \|R_1\|) = \|e + \alpha a\| = 1 + \|R_1\| + R_3$ where

$R_3 = 0(\alpha^2)$. Combining these inequalities we get

$$\|e + \alpha a\| - 1 + 0(\alpha^2) \leq \log\|\exp(\alpha a)\| \leq \|e + \alpha a\| - 1 + 0(\alpha^2)$$

multiplying throughout by α^{-1} and taking the limit as α approaches 0 from above, gives the result. //

.3.2.1. COROLLARY. $\lim_{\alpha\to 0+} \alpha^{-1}\{\|e + \alpha a\| - 1\} = \sup_{\alpha>0} \alpha^{-1}\log\|\exp(\alpha a)\|$.

Proof. Clearly $\lim_{\alpha\to 0+} \alpha^{-1}\log\|\exp(\alpha a)\| \leq \sup_{\alpha>0} \alpha^{-1}\log\|\exp(\alpha a)\|$

Conversely for any $\alpha>0$

$$(2\alpha)^{-1}\log \|\exp(2\alpha a)\| = (2\alpha)^{-1}\log \|(\exp(\alpha a))^2\|$$

$$\leq (2\alpha)^{-1}\log \|\exp(\alpha a)\|^2,$$ since log monotonically increasing, therefore

$$(2\alpha)^{-1}\log \|\exp(2\alpha a)\| = \alpha^{-1}\log \|\exp(\alpha a)\| .$$

Hence, since $\alpha^{-1} \log \|\exp(\alpha a)\|$ is a continuous function of α it is a monotonically decreasing function of α and so the reverse inequality is true. That is

$$\lim_{\alpha\to 0+} \alpha^{-1}\log \|\exp(\alpha a)\| = \sup_{\alpha>0} \alpha^{-1}\log \|\exp(\alpha a)\| .$$

The result now follows from Lemma .3.2. //

.3.2.2. **COROLLARY.** $\operatorname{Sup} \operatorname{Re} V(a) = \lim_{\alpha\to 0+} \alpha^{-1}\log \|\exp(\alpha a)\|$

$$= \sup_{\alpha>0} \alpha^{-1}\log \|\exp(\alpha a)\| .$$

Proof. This follows directly from the above corollary and Corollary .3.1.2. together with Lemma .3.2. //

.3.3. **DEFINITION.** For a Banach Algebra A the *numerical radius* of $a \in A$ is

$$v(a) = \sup \{|\lambda| : \lambda \in V(a)\}.$$

Clearly $v(a) \leq \|a\|$.

If A is an operator algebra, $A = B(E)$ for some normed linear space E, then by Corollary .3.1.3. it is clear that

$$v(a) = \sup\{|\lambda| : \lambda \in W(a)\}$$

$$= \sup\{|\lambda| : \lambda \in W_\phi(a)\},$$ for any support mapping ϕ of E into E'.

Since $V(\lambda a) = \lambda V(a)$ for any $\lambda \varepsilon C$ it is clear that

$$v(a) = \sup_{|\lambda|=1} \sup \operatorname{Re} V(\lambda a).$$

Hence the previous two lemmas give several expressions for $v(a)$. For easy reference we collect these together in the following theorem.

3.4. **THEOREM.** *For a unital Banach Algebra* A *and* $a \varepsilon A$ *the following are equal to* $v(a)$:

i) $\sup_{|\lambda|=1} \lim_{\alpha\to 0+} \alpha^{-1}\{\|e+\alpha\lambda a\| - 1\}$

ii) $\sup_{|\lambda|=1} \lim_{\alpha\to 0+} \alpha^{-1} \log \|\exp(\alpha\lambda a)\|$

iii) $\sup_{0\neq\lambda\varepsilon C} |\lambda|^{-1} \log \|\exp(\lambda a)\|$.

Proof. By the remark preceding the theorem, i) follows from Corollary 3.1.2, ii) from Corollary 3.2.2 and, noting that

$$\sup_{|\lambda|=1} \sup_{\alpha>0} \alpha^{-1} \log \|\exp(\alpha\lambda a)\| = \sup_{0\neq\lambda\varepsilon C} |\lambda|^{-1} \log \|\exp(\lambda a)\|,$$

iii) follows from Corollary 3.3.2. //

For a Banach Algebra A, $\lambda \varepsilon C$ and $a,b \varepsilon A$ we have already seen that $V(\lambda a) = \lambda V(a)$ and $V(a+b) \subseteq V(a) + V(b)$. It is clear from these and the definition of $v(a)$ that

$$v(\lambda a) = |\lambda|\, v(a)$$

and $v(a+b) \leq v(a) + v(b)$.

Thus $v(\cdot)$ is a linear pseudonorm on A. The next Theorem, due essentially to Bohnenblust and Karlin [1] shows that $v(\cdot)$ is a norm on A, regarded as a linear space, equivalent to $\|\cdot\|$.

Hence in particular $v(a) = 0$ implies $a = 0$.

3.5. **THEOREM**; *If* A *is a unital Banach Algebra, then for* $a \in A$, $\frac{1}{e} \|a\| \leq v(a)$.

Proof. Since $v(a) = \sup\limits_{0 \neq \lambda \in C} |\lambda|^{-1} \log \|\exp(\lambda a)\|$, by Theorem 3.4 iii), we have $e^{|\lambda| v(a)} \geq \|\exp(\lambda a)\|$ for all $0 \neq \lambda \in C$.

Therefore
$$\|a\| = \frac{1}{2\pi} \left\| \oint_{|\lambda|=R} \frac{\exp(\lambda a)}{\lambda^2} d\lambda \right\|, \quad R > 0$$
$$= \frac{1}{2\pi} \left\| \int_0^{2\pi} \frac{\exp(\lambda a)}{\lambda} d\theta \right\| \text{ where } \lambda = Re^{i\theta}$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} \left\| \frac{\exp(\lambda a)}{R} \right\| d\theta$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} R^{-1} e^{Rv(a)} d\theta$$
$$= R^{-1} e^{Rv(a)}$$

Thus $\|a\| \leq \min\limits_{R>0} R^{-1} e^{Rv(a)}$. Differentiation shows the minimum occurs at $R = v(a)^{-1}$.

Therefore $\|a\| \leq v(a)e^{-1}$

or $\frac{1}{e}\|a\| \leq v(a)$. //

B.W. Glickfeld [2] has constructed a Banach Space X and operator $T \in B(X)$ for which $v(T) = \frac{1}{e}\|T\|$ showing that the inequality in the previous Theorem is best possible in general. The less sharp inequality $\frac{1}{4}\|a\| \leq v(a)$ was proved by Lumer [1] but a more elementary proof is given by Bonsall [1]. For many purposes this last inequality is sufficient. It is interesting to note that when $A = B(H)$ for some Hilbert space H, then the constant $\frac{1}{e}$ can be replaced by $\frac{1}{2}$ which is also best possible as the operator $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ over ℓ_2^2 shows.

For a unital Banach Algebra A and $a \in A$ we have determined a number of expressions for the supremum of the real part of $V(a)$. These will be of importance in our later theory. We have also used these bounds to prove that when $A = B(E)$, for E a normed linear space, the numerical ranges $W(T)$ and $W_\phi(T)$, for any support mapping ϕ of E into E', share a common closed convex hull which we have identified as $V(T)$. We were thus able to define the numerical radius of $a \in A$ independently of the particular type of numerical range being considered. The numerical radius was then shown to be an equivalent linear space norm for a unital Banach Algebra.

1.4. RELATION BETWEEN THE NUMERICAL RANGES AND THE SPECTRUM

It is well known that, for a Hilbert space H and $T \in B(H)$, $\sigma(T) \subseteq \overline{W(T)}$ [Halmos, 1]. The corresponding result for the algebra numerical range of an element of a unital normed algebra is simply established [J. G. Stampfli and J. P. Williams, 1].

4.1. **THEOREM.** *For a unital normed algebra* A *and* $a \in A$ *we have* $\sigma(a) \subseteq V(a)$.

Proof. Let $C_m(a)$ be a maximal commutative subalgebra of A containing a and e. Since $\sigma_{C_m(a)}(b) = \sigma_A(b)$ for $b \in C_m(a)$, the Gelfand theory shows that for any $\lambda \in \sigma(a)$ there exists a non-zero multiplicative functional f in $C_m(a)'$ such that $f(a) = \lambda$. In particular $\|f\| = 1$ and $f(e) = 1$. So by the Hahn-Banach Theorem f can be extended to $\underline{f} \in A'$ such that $\underline{f} \in D(e)$ and

$\lambda = \underline{f}(a) \in V(a)$ by lemma 2.5. //

A similar result for the spatial numerical range is not so easily established. Clearly if λ is an eigenvalue of $T \in B(E)$, where E is a normed linear space, and x, $\|x\| = 1$, the corresponding eigenvector, then $\lambda = f(Tx) \in W(T)$ where $f \in D(x)$. If λ is an approximate eigenvalue of T then there exists x_n, $\|x_n\| = 1$, such that $\|Tx_n - \lambda I x_n\| \to 0$ and so

$$|f_n(Tx_n) - \lambda| \leq \|Tx_n - \lambda I x_n\| \to 0 \text{ where } f_n \in D(x_n).$$

Hence $\lambda \in \overline{W(T)}$. The full result that $\sigma(T) \subseteq \overline{W(T)}$ was proved by

Williams [1] using the subreflexivity of Banach spaces established by Bishop and Phelps [1].

.4.2. THEOREM. *If* X *is a Banach Space and* $T \in B(X)$ *then* $\sigma(T) \subseteq \overline{W(T)}$

Proof. For $\lambda \notin \overline{W(T)}$, $d(\lambda, \overline{W(T)}) = \inf\{|\lambda - \xi| : \xi \in \overline{W(T)}\} = d > 0$ hence for $x \in X$ let $\underline{x} = x/\|x\|$ then

$$\begin{aligned} \|[T - \lambda I]\underline{x}\| &\geq |f([T - \lambda I]\underline{x})| \text{ for all } f \in D(\underline{x}) \\ &= |f(T\underline{x}) - \lambda| \\ &\geq d. \end{aligned}$$

So $\|[T - \lambda I]x\| \geq d\|x\|$ for all $x \in X$.

This implies $(T - \lambda I)^{-1}$ exists and is bounded on the Range of $(T - \lambda I)$, $R(T - \lambda I)$, [Bachman and Narici, 1, p. 241] and that $R(T - \lambda I)$ is closed.

We now show $R(T - \lambda I)$ is dense in X and so $R(T - \lambda I) = X$ and hence $(T - \lambda I)^{-1} \in B(X)$ so $\lambda \notin \sigma(T)$ as required.

To see that $R(T - \lambda I)$ is dense in X it is sufficient to prove that

$$N(T' - \lambda I') = \{f \in X' : (T - \lambda I)'f = 0\} = \{0\}$$

and since the set of functionals which attain their norm in X' are dense in X' [Bishop and Phelps, 1] it is sufficient to prove that $(T - \lambda I)'f \neq 0$ when f attains its norm.

Hence let $f \in X'$, $\|f\| = 1$, attain its norm at $x \in X$

then $\|(T - \lambda I)'\| \geq \|(T - \lambda I)'f\|$

$$\geq |(T - \lambda I)'f(x)|$$

$$= |f(Tx) - \lambda|$$

$$\geq d, \text{ as before.}$$

So $(T - \lambda I)'f \neq 0$ as required. //

If A is a unital Banach algebra and $a, b \in A$ such that $0 \notin V(a)$ then, by Theorem .4.1, a^{-1} exists and we have

$$\sigma(a^{-1}b) \subseteq \{\lambda/\xi : \lambda = f(b) \text{ and } \xi = f(a),\ f \in D(e)\}$$

since $a^{-1}b - \lambda e = a^{-1}(b - \lambda a)$ so if $\lambda \in \sigma(a^{-1}b)$ then $0 \in \sigma(b - \lambda a) \subseteq V(b - \lambda a)$. Hence there exists $f \in D(e)$ such that $0 = f(b) - \lambda f(a)$ or $\lambda = f(b)/f(a)$.

Williams [1] used Theorem .4.2. to obtain a similar result for spatial numerical ranges. Namely, if X is a Banach Space and $T, V \in B(X)$ such that $0 \notin W(T)$ then

$$\sigma(T^{-1}V) \subseteq \overline{W(V)} \,/\, \overline{W(T)} = \{\lambda/\xi = \lambda \in \overline{W(V)},\ \xi \in \overline{W(T)}\}.$$

The previous Theorem (.4.2.) is considerably strengthened in a recent result of M. J. Crabb [Thesis, 2] which shows $\overline{\text{co}}\ \sigma(T) \subseteq \overline{W(T)}$ for T an operator over a complex Banach space. Although this result is not necessary for our development of the theory of numerical ranges, we will indicate some of its implications as they arise.

We now proceed to approximate the spectrum more closely by numerical ranges. The results were indicated by a formula for the spectral radius obtained by Bohnenblust and Karlin [1], for a commutative normed algebra, and Bonsall [2] for any Banach algebra. As we will show they also give, as a special case, a result of J. P. Williams [2] for operators on a Hilbert space. A similar development of these results has also been given by Bonsall and Duncan [1].

We first need a technical lemma from which we obtain several useful corollaries. The lemma generalizes a step of William's proof [2] and Bonsall and Duncans' [1, 2.10].

4.3. LEMMA. *If associated with the element* a *of a unital Banach algebra we have a family of closed subsets of the complex plane* $\{W_\lambda(a)\}_{\lambda \in \Lambda}$ *such that:*

i) $\overline{co}\,\sigma(a) \subseteq W_\lambda(a)$ *for all* $\lambda \in \Lambda$;

ii) $W_\lambda(\alpha a + \beta e) = \alpha W_\lambda(a) + \beta$ *for all* $\alpha, \beta \in C$ *and* $\lambda \in \Lambda$;

iii) If $\rho(\alpha a + \beta e) < 1$ *then for some* $\lambda \in \Lambda$

$$\sup\{|\xi| : \xi \in W_\lambda(\alpha a + \beta e)\} \leqslant 1;$$

then $\overline{co}\sigma(a) = \bigcap_{\lambda \in \Lambda} W_\lambda(a)$.

Proof. By i) $\overline{co}\,\sigma(a) \subseteq \bigcap_{\lambda \in \Lambda} W_\lambda(a)$.

If $B = \{\xi \in C: |\xi - \beta| \leqslant r\}$ is a disk such that $\sigma(a) \subseteq \text{Int } B$

then $r^{-1}(a - \beta e)$ is such that $\rho(r^{-1}(a - \beta e)) < 1$ so by iii) there exists $\lambda \in \Lambda$ such that

$$\sup\{|\xi| : \xi \in W_\lambda(r^{-1}(a - \beta e))\} \leq 1$$

therefore $W_\lambda(r^{-1}(a - \beta e)) \subseteq \{\xi : |\xi| \leq 1\}$

and so by ii)

$$W_\lambda(a) \subseteq B.$$

Since the intersection of all such disks, B, equals $\overline{\text{co}}\ \sigma(a)$ it follows that

$$\bigcap_{\lambda \in \Lambda} W_\lambda(a) \subseteq \overline{\text{co}}\ \sigma(a)$$

proving the result. //

Theorem .4.1. shows that the algebra numerical range of $a \in A$ satisfies all the conditions of Lemma .4.3. with the possible exception of iii) which requires that $v(\alpha a + \beta e) \leq 1$ whenever $\rho(\alpha a + \beta e) < 1$. Further if $A = B(X)$, where X is a Banach space, Crabb's result shows the same is true for the closure of the spatial numerical range.

.4.3.1. **Corollary.** i) *For a unital Banach Algebra* A, *if* $a \in A$ *is such that* $\rho(a + \alpha e) = v(a + \alpha e)$ *for all* $\alpha \in C$, *then* $\overline{\text{co}}\sigma(a) = V(a)$

ii) *For a Banach Space* X, *if* $T \in B(X)$ *is such that* $\rho(T + \alpha I) = v(T + \alpha I)$ *for all* $\alpha \in C$, *then* $\overline{W(T)}$ *is convex and equal to* $\overline{\text{co}}\ \sigma(T)$.

Proof. By Theorem .4.1. we may take $\{W_\lambda(a)\}_{\lambda \in \Lambda}$ of Lemma .4.3. as $V(a)$ and iii) is satisfied since if $\rho(\alpha a + \beta e) < 1$ then

$$\begin{aligned} v(\alpha a + \beta e) &= |\alpha| v(a + \alpha^{-1}\beta e) \\ &= |\alpha| \rho(a + \alpha^{-1}\beta e) \\ &= \rho(\alpha a + \beta e) \\ &< 1. \end{aligned}$$

ii) Follows similarly to i) replacing Theorem .4.1. by Crabb's result [2]. //

For a Banach algebra A and $a \in A$ we now proceed to construct a family $\{W_\lambda(a)\}_{\lambda \in \Lambda}$ such as required in Lemma .4.3. If p is an equivalent norm on A we can define the numerical range of a, $V_p(a)$, in the Banach algebra A with norm p. We will see that the family of all such numerical ranges satisfies the conditions of lemma .4.3. To ensure that iii) of Lemma .4.3. is satisfied it is sufficient to show that if $a \in A$ is such that $\rho(a) < 1$ then there exists an equivalent norm, p, on A for which a is a contraction, that is $p(a) \leqslant 1$, for then $v_p(a) \leqslant p(a) \leqslant 1$, where $v_p(a)$ denotes the numerical radius of $V_p(a)$.

Before proceeding we require the well known result that if $\rho(a) < 1$, then the iterates of a are uniformly bounded [G.-C. Rota, 1].

.4.4. **LEMMA.** *For a Banach algebra* A *if* $a \in A$ *is such that* $\rho(a) = r < 1$, *then there exists* $M > 0$ *such that* $\Sigma \|a^n\|^2 = M^2$, *and so* $\|a^n\| \leq M$ *for all* n.

Proof. Define $f : C \to C$ by $f(z) = \sum_{n=0}^{\infty} \|a^n\| z^n$, which, by the spectral radius formula, converges for $|z| < r^{-1}$. Thus $f(z)$ is analytic for $|z| < r^{-1}$ and so

$$I = \oint_{|z|=r} f(z)\overline{f(z)}dz < \infty \text{ for } R < r^{-1}.$$

Using Parseval's Identity we have $2\pi I = \sum_{n=0}^{\infty} \|a^n\|^2 R^{2n}$ so choosing $R = 1 < r^{-1}$, as $r < 1$, we have

$$M^2 = \Sigma \|a^n\|^2 < \infty.$$

//

Let the unital Banach algebra A be a closed subalgebra of B(X), for some Banach space X, with the unit e acting as the identity operator on X, Then for a given $a \in A$, with $\rho(a) < 1$, and $x \in X$ define:

$$p_a(x) = \sup_{n=0}^{\infty} \{\|a^n x\|\}$$

$$q_a(x) = \left[\sum_{n=0}^{\infty} \|a^n x\|^2\right]^{\frac{1}{2}}$$

$r_a(x) = \inf \sum_{k=0}^{n} \|x_k\|$, where the infimum is taken over all possible finite sets $x_1, \ldots x_n$ ($x_k \in X$) such that

$$x = \sum_{k=0}^{n} a^k x_k.$$

Let $P_a(b) = \sup\{p_a(bx) : p_a(x) = 1\}$ for $b \in A$ and similarly define $Q_a(b)$ and $R_a(b)$.

If we take $X = A$ and regard A as a subalgebra of $B(X)$ under the left regular representation, then if A is commutative Bohnenblust and Karlin [1] prove that $r_a(\cdot)$ is an equivalent algebra norm on A with $r_a(a) \leq 1$, while Bonsall [2] shows that $r_a(\cdot)$ is always an equivalent linear space norm on A and the corresponding operator norm $R_a(\cdot)$ on A is such that $R_a(a) < 1$.

We prove a similar result to Bonsall's for $p_a(\cdot)$ and $q_a(\cdot)$. Bonsall and Duncan [1] also use $p_a(\cdot)$.

.4.5. **LEMMA.** *Let the unital Banach algebra* A *be a subalgebra of* $B(X)$, X *some Banach space, with* $e \equiv I$, *then for* $a \in A$, *with* $\rho(a) < 1$, $p_a(\cdot)$ *and* $q_a(\cdot)$ *are equivalent norms on* X *and the operator norms* $P_a(\cdot)$ *and* $Q_a(\cdot)$ *are equivalent norms on* A *with* $P_a(a)$, $Q_a(a) \leq 1$.

Proof. If $P_a(x) = 0$ then $\|a^0 x\| = \|x\| = 0$ so $x = 0$.

Also
$$P_a(\alpha x) = \sup\{\|a^n \alpha x\|\} = |\alpha| \sup\{\|a^n x\|\} = |\alpha| P_a(x), \text{ and}$$

$$P_a(x + y) = \sup\{\|a^n(x + y)\|\} \leq \sup\{\|a^n x\|\} + \sup\{\|a^n y\|\} = P_a(x) + P_a(y)$$

so $P_a(\cdot)$ is a norm on X. Further

$P_a(x) \geq \|a^0 x\| = \|x\|$ and, since $\|a^n x\| \leq M\|x\|$ by lemma .4.4., $p_a(x) \leq M\|x\|$, so $P_a(\cdot)$ is an equivalent norm on X.

Lastly
$$P_a(a) = \sup_{n=0}^{\infty}\{\|a^n(ax)\| : \sup\|a^n x\| = 1\} = \sup_{n=1}^{\infty}\{\|a^n x\|\} \leq \sup_{n=0}^{\infty}\{\|a^n x\|\} = 1.$$

Using Minkowski's inequality the result for $q_a(\cdot)$ and $Q_a(\cdot)$ follows along similar lines. //

4.5.1. **Corollary.** *For a unital Banach algebra* A *and* $a \in A$ *with* $\rho(a) < 1$, *there exists an equivalent norm* p *on* A *with which* A *is again a unital Banach algebra and* $p(a) \leq 1$.

Proof. In the last lemma take $X = A$ and regard A as a subalgebra of $B(X)$ under the left regular representation, then for $p(a) = P_a(a)$ (or $Q_a(a)$) we see that $p(a) \leq 1$ and since p is the algebra norm on A arising from the renorming of X by $p_a(\circ)$, and since $e \equiv I$, we have $p(e) = 1$ and so A with norm p is a unital Banach algebra. //

.4.5.2. **Corollary.** *For a Banach Space* X *and* $T \in B(X)$ *with* $\rho(T) < 1$, *there exists an equivalent norm* p *on* X *such that*

$$p(T) = \sup\{p(Tx) : x \in X, p(x) = 1\} \leq 1.$$

Proof. Let $A = B(X)$ and apply Lemma .4.5. with $p = p_a(\circ)$ (or $q_a(\circ)$). //

4.5.3. **Corollary.** *For the Banach space* X, $T \in B(X)$ *is an isometry for an equivalent norm on* X *if* $\|T^n\| \leq M > 0$, *for all* n, *and for each* $x \in X$, $\|x\| \leq \|T^n x\|$ *for some* $n > 0$.

Proof. Since $\|T^n\| \leq M$ for all n, we see, by the proof of Lemma .4.5., that for each $x \in X$

$$p(x) = \sup_{n=0}^{\infty} \{\|T^n x\|\} \text{ is an equivalent norm on } X.$$

Further
$$p(Tx) = \sup_{n=1}^{\infty} \{\|T^n x\|\} = \sup_{n=0}^{\infty} \{\|T^n x\|\} = p(x)$$

since the condition $\|x\| \leq \|T^n x\|$, for some $n > 0$, insures that the latter supremum is not attained for $n = 0$. So $p(Tx) = p(x)$ for all $x \in X$ and T is an isometry on X for the equivalent norm p. //

A simple calculation shows the last Corollary is a special case ($d = 1/M$) of a result by D. Koehler and P. Rosenthal [1, Theorem 2], who show that $T \in B(X)$ is an isometry for an equivalent norm if and only if there exist $d, M > 0$ such that

$\|T^n\| \leq M$ and $d\|x\| \leq \|T^n x\|$, for all n and all $x \in X$.

We can now prove the main result. It also appears in Bonsall and Duncan [1], while the corollary was first proved by Bohnenblust and Karlin [1] for a commutative algebra and later by Bonsall [2] for arbitrary unital Banach algebras.

.4.6. THEOREM. i) *For a unital Banach Algebra* A *and* $a \in A$ *we have*

$\overline{co}\,\sigma(a) = \bigcap V_p(a)$ *where the intersection is taken over all equivalent norms* p *on* A *with* $p(e) = 1$.

ii) *For a Banach space* X *and* $T \in B(X)$ *we have*

$\overline{co}\,\sigma(T) = \bigcap Vp(T)$

$= \bigcap \overline{Wp(T)}$ *where the intersection is taken over all equivalent norms* p *on* X.

Proof. i) Since $\sigma(a)$ is unaltered by equivalent renorming we have $\overline{co}\,\sigma(a) \subseteq V_p(a)$ for all p. Further if $b = \alpha a + \beta e$ is such that

$\rho(b) < 1$ then by Corollary .4.5.1. there exists an equivalent norm p such that $v_p(b) \leq 1$. So the family $V_p(a)$ satisfies the requirements of Lemma .4.3. and the result follows.

ii) From Crabb's result [2] $\overline{co}\ \sigma(T) \subseteq \overline{W_p(T)}$ for all p. The result now follows by arguments exactly similar to those used in i) with Corollary .4.5.1. replaced by Corollary .4.5.2.//

.4.6.1. **Corollary.** *For a unital Banach algebra* A *and* $a \in A$ *we have*

$\rho(a) = \inf v_p(a)$ *where the infimum is taken over all equivalent norms* p *on* A *such that* $p(e) = 1$.

Proof. This proof is immediate from Theorem .4.6 i). //

From the proof of Theorem .4.6. and Corollary .4.5.1. we see that the intersection in Theorem 4.6. need not be taken over all equivalent norms but only over the family of equivalent norms $P_s(\circ)$ (or $Q_s(\circ)$) where $s \in A$, $\rho(s) < 1$. While this is of little importance for general Banach algebras, it does allow the result to be sharpened for $A = B(H)$ where H is some Hilbert space.

Using a result of Rota [1], Williams [2] has proved that for an operator T over a Hilbert space H

$$\overline{co}\ \sigma(T) = \bigcap_S W(S^{-1}TS)$$

where S is a bounded one to one operator from H onto a closed subspace of H, with bounded inverse S^{-1}.

We prove that

$$\overline{\text{co}}\ \sigma(T) = \bigcap_S (W(S^{-1}TS)) \text{ where } S \text{ is a regular element of } B(H).$$

Our proof shows how this, and William's result, arise as special cases of the theory developed so far.

.4.7. LEMMA. *If* H *is a Hilbert space and* $T \in B(H)$ *has* $\rho(T) < 1$, *then there exists a regular* $S \in B(H)$ *such that the equivalent renorming* $q_T(\cdot)$ *arises from the original norm by* $q_T(x) = \|Sx\|$ *for all* $x \in H$.

Proof. It is readily seen that a second inner-product is defined on H by

$$(x, y)_T = \sum_{n=0}^{\infty} (T^n x, T^n y) \text{ for all } x, y \in H.$$

Letting H_T denote H with inner-product $(\ , \ )_T$ we see that $\|x\|_T^2 = (x, x)_T = q_T(x)^2$ and so H_T is a Hilbert space.
It will now be sufficient to show that H and H_T have the same orthogonal dimension, for then by [Wilansky, 1, p. 130, Theorem 4] there exists an isometric isomorphism S of H_T onto H, and since by Lemma 4.5., H_T is H equivalently renormed we may identify S with a regular operator on H such that $\|Sx\| = q_T(x)$ for all $x \in H$.
Now that H, H_T have the same orthogonal dimension follows by adapting a standard argument.

Let B, B_T be maximal orthonormal sets in H, H_T respectively.
For each $b \in B$ let

$B_T(b) = \{c \in B_T : (c, b)_T \neq 0\}$, then $B_T(b)$ is a countable set. Now assume there exists $c \neq 0$, $c \in B_T$ but $c \notin B_T(b)$ for any $b \in B$, then $(c,b) = 0$ for all $b \in B$, but H is a Hilbert space and so every element of H can be written as a countable linear sum of elements in B. Therefore $(c,y) = 0$ for all $y \in H$ and so $c = 0$. Hence every $c \in B_T$ is in some $B_T(b)$. That is

$B_T \subseteq \bigcup_{b \in B} B_T(b)$ and $\# B_T \leq \# B$. $N_0 = \# B$.

Similarly $\# B \leq \# B_T$ and so $\# B = \# B_T$. //

..4.7.1. **Corollary.** *If* H *is a Hilbert space and* $T \in B(H)$ *has* $\rho(T) < 1$, *then there exists regular* $S \in B(H)$ *such that for all* $V \in B(H)$

$\|SVS^{-1}\| = Q_T(V)$. *So in particular* $\|STS^{-1}\| \leq 1$.

Proof. By Lemma .4.7.

$$
\begin{aligned}
Q_T(V) &= \sup\{q_T(Vx) : x \in H,\ q_T(x) = 1\} \\
&= \sup\{\|SVx\| : x \in H, \|Sx\| = 1\} \\
&= \sup\{\|SVS^{-1}y\| : y \in H,\ \|y\| = 1\} \\
&= \|SVS^{-1}\| \text{, since } S \text{ is regular and so each } x \in H \text{ can}
\end{aligned}
$$

be written $x = S^{-1}y$ some $y \in H$. That $\|STS^{-1}\| \leq 1$ follows from Lemma .4.5. //

.4.8. **THEOREM.** *For a Hilbert space* H *and* $T \in B(H)$ *we have*

$$\overline{co}\,\sigma(T) = \bigcap W(STS^{-1})$$ *where the intersection is taken over all regular* $S \in B(H)$.

Proof. Since $\sigma(STS^{-1}) = \sigma(T)$ for all regular $S \in B(H)$ and since $W(V)$ is convex for all $V \in B(H)$ we see that $\overline{co}\,\sigma(T) \subseteq \overline{W(STS^{-1})}$. Further if $V = \alpha T + \beta I$ is such that $\rho(V) < 1$ then by corollary .4.7.2. there exists regular $S \in B(H)$ such that

$$v(SVS^{-1}) \leqslant \|SVS^{-1}\| \leqslant 1.$$

So the family $W(STS^{-1})$ satisfies the conditions of Lemma .4.3. and the result follows. //

.4.8.1. **Corollary.** *For a Hilbert Space* H *and* $T \in B(H)$ *we have*

$$\rho(T) = \inf v(STS^{-1})$$ *where the infimum is taken over all regular* $S \in B(H)$.

That Theorem .4.8. is not true for arbitrary Banach spaces is shown by Crabb, Duncan and C.M. McGregor [1], who show that for a finite dimensional Banach space X and $T \in B(X)$

$$\rho(T) = \inf_{S} v(STS^{-1})$$

if and only if the norm of X is absolute with respect to some basis.

In this section we have seen that the spectrum of an element of a unital Banach algebra is contained in the closure of the numerical range of that element. Since $\overline{co}\,\sigma(a) \subseteq V(a)$ the closest relation between the spectrum and numerical range would be $\overline{\text{co}}\ \sigma(a) = V(a)$. We have shown this to be the case if a is such that $\rho(a+\lambda e) = v(a+\lambda e)$ for all $\lambda \varepsilon C$. We also saw that $\overline{\text{co}}\ \sigma(a)$ may be approximated arbitrarily closely by numerical ranges defined with respect to equivalent norms. Further relations between the spectrum and numerical range for particular types of operators will be developed later in Chapter 3.

5. HERMITIAN ELEMENTS

In this section we make a study of those elements of a unital Banach Algebra distinguished by having real numerical ranges. Such elements are fundamental to the study of Banach Algebras, through numerical range, and are the basis of the theory developed in Chapter 2.

Although the proofs of many of the results in this section are well known we give them because of their importance and the constant use we will make of the results in later theory.

5.1. DEFINITION. The element a of the Banach Algebra A is *hermitian* if V(a) is real.

If A = B(E) for some normed linear space E, then Corollary 3.1.3 shows that it does not matter which numerical range we use to define hermitian, for if one is real then all are.

The following Lemma gives a simple and useful criterion for determining hermitian elements, and as we will show later, for the right choice of functions, gives a characterisation of hermitian elements.

5.2. LEMMA. *For a unital Banach Algebra* A, $a \in A$ *is hermitian if, for some pseudonorm* p *on the real linear space generated by* e, a *and* a^2 *with* $p(e) = 1$, *we have*

$$[v(a+\lambda e)]^2 \leq p[(a+\lambda e)(a+\bar{\lambda} e)] \text{ for all } \lambda \in C.$$

Proof. For $f \in D(e)$ let $f(a) = \alpha+i\beta$ α,β real then for any real γ

$$\begin{aligned}
\gamma^2+2\gamma\beta+\beta^2 &= (\gamma+\beta)^2 \\
&= |i\gamma+i\beta|^2 \\
&= |f(a-\alpha e+i\gamma e)|^2 \\
&\leq [v(a+(-\alpha+i\gamma)e)]^2 \\
&\leq p[(a+(-\alpha+i\gamma)e)(a+(-\alpha-i\gamma)e)] \\
&= p[(a-\alpha e)^2+\gamma^2 e] \\
&\leq p[(a-\alpha e)^2]+\gamma^2.
\end{aligned}$$

So $2\gamma\beta+\beta^2 \leq p[(a-\alpha e)^2]$ for all real γ.

But this is only possible if $\beta = 0$, therefore $f(a) = \alpha$ is real and so by Lemma 2.5 $v(a)$ is real and a hermitian. //

Note that the norm of A, the numerical radius and the spectral radius, restricted to the real linear space generated by e, a and a^2 are all suitable choices for p in the above Lemma. We will later see that the above Lemma gives a necessary as well as sufficient condition for a to be hermitian.

The next lemma uses the results of Section 3 to obtain several characterisations of hermitian elements. These characterisations were first used by Lumer [1,2].

5.3. LEMMA. *For a unital Banach Algebra* A *and* $a \in A$ *the following are equivalent to* a *being hermitian:*

i) $\lim_{\alpha\to 0} \alpha^{-1}\{\|e+i\alpha a\| -1\} = 0$ (α *real*)

ii) $\|\exp(i\alpha a)\| = 1$ *for all real* α.

Proof. i) If V(a) is real, then

$$\sup \operatorname{Re} V(ia) = \sup \operatorname{Re} V(-ia) = 0.$$

So by Corollary 3.1.2

$$\lim_{\beta\to 0+} \beta^{-1}\{\|e\pm i\beta a\| -1\} = 0$$

or

$$\lim_{\alpha\to 0} \alpha^{-1}\{\|e+i\alpha a\| -1\} = 0 \quad (\alpha \text{ real}).$$

ii) From Corollary 3.2.1

$$\sup_{\beta>0} \beta^{-1} \log \|\exp(\pm i\beta a)\| = \lim_{\beta\to 0+} \beta^{-1}\{\|e\pm i\beta a\| -1\} = 0,$$

by the proof of i) if V(a) is real.

Therefore, for all $\beta > 0$

$$\log \|\exp(\pm i\beta a)\| = 0$$

or

$$\|\exp(i\alpha a)\| = 1 \quad \text{for all real } \alpha. \qquad //$$

The following Lemma, essentially due to I. Vidav [1], is one of the most important in the theory of numerical range and provides the basis for much of our later work. A proof avoiding an appeal to Polya and Szegö is given by Bonsall and Duncan [1].

5.4. LEMMA. *For a unital Banach Algebra* A *and hermitian element* $a \in A$ *we have* $\overline{\mathrm{co}}\,\sigma(a) = V(a)$ *and so in particular* $\rho(a) = v(a)$.

Proof. Since $\sigma(a) \subseteq V(a)$ is real, it is enough to show $\sup \sigma(a) = \sup V(a)$ and $\inf \sigma(a) = \inf V(a)$.

Further since $\inf \sigma(a) = -\sup \sigma(-a)$ and

$$\inf V(a) = -\sup V(-a)$$

it is sufficient to show $\sup \sigma(a) = \sup V(a)$ for any hermitian $a \in A$. The first step in the proof is to obtain two expressions, similar to those of Corollary 3.2.2, for $\sup \sigma(a)$. The argument we use is given by Stampfli and Williams [1].

Let $\gamma = \sup \sigma(a)$, then since exp is an increasing function on R, $e^{\gamma} = \sup\{e^{\alpha} : \alpha \in \sigma(a)\}$, but by the spectral mapping theorem $\sigma(\exp(a)) = \{e^{\alpha} : \alpha \in \sigma(a)\} \subseteq [0,\infty)$ therefore $e^{\gamma} = \rho(\exp(a))$.

Applying the spectral radius formula gives

$$\begin{aligned} e^{\gamma} &= \lim_{n\to\infty} \|\exp(na)\|^{1/n} \\ &= \inf_{n>0} \|\exp(na)\|^{1/n}. \end{aligned}$$

Therefore since log is a continuous function we have

$$\gamma = \lim_{\alpha\to\infty} \alpha^{-1} \log \|\exp(\alpha a)\|$$
$$= \inf_{\alpha>0} \alpha^{-1} \log \|\exp(\alpha a)\| .$$

The latter expression shows that $e^{\alpha\gamma} \leq \|\exp(\alpha a)\|$ for all $\alpha \geq 0$.

While the first expression gives $0 = \lim_{\alpha\to\infty} \alpha^{-1} \log(\|\exp(\alpha a)\| e^{-\alpha\gamma})$.

So $\log(\|\exp(\alpha a)\| e^{-\alpha\gamma}) = o(\alpha)$ as $\alpha \to \infty$.

Now for any $f \in A'$ consider the function $\theta : C \to C$ defined by

$$\theta(\lambda) = f(\exp(\lambda(a-\gamma e))) \quad \text{for all } \lambda \in C.$$

It is easily seen from the definition of exp that θ is analytic in any bounded region of the complex plane.

Also for $\varepsilon > 0$ and $\lambda = \alpha+i\beta$ α,β real, we have

$$|\theta(\lambda)e^{-\varepsilon\lambda}| = |f(\exp(\alpha(a-\gamma e))\exp(i\beta(a-\gamma e)))|\ |e^{-\varepsilon\alpha}|$$
$$\leq \|\exp(\alpha(a-\gamma e))\|\ |e^{-\varepsilon\alpha}|,$$

since $a-\gamma e$ is hermitian and so by Lemma 5.3ii) $\|\exp(i\beta(a-\gamma e))\| = 1$.

Therefore, for α sufficiently large

$$|\theta(\lambda)e^{-\varepsilon\lambda}| \leq (\|\exp(\alpha a)\| e^{-\alpha\gamma})\ |e^{-\varepsilon\alpha}|$$
$$= e^{o(\alpha)}\ e^{-\varepsilon\alpha}$$

and hence $\lim_{\alpha\to\infty} |\theta(\lambda)e^{-\varepsilon\lambda}| = 0$.

So by Polya and Szegö [1,p.147] we have $|\theta(\lambda)| \leq 1$ for Re $\lambda \geq 0$.

That is $|f(\exp(\lambda(a-\gamma e)))| \leq 1$ for Re $\lambda \geq 0$, but f was arbitrary so $\|\exp(\lambda(a-\gamma e))\| \leq 1$ for Re $\lambda \geq 0$ and so in particular for $\alpha > 0$

$$\|\exp(\alpha(a-\gamma e))\| \leq 1$$

or $\|\exp(\alpha a)\| \leq e^{\alpha\gamma}$.

Combining this with the reverse inequality previously established we have

$$\|\exp(\alpha a)\| = e^{\alpha\gamma}.$$

Now by Corollary 3.2.2

$$\begin{aligned}\sup V(a) &= \sup_{\alpha>0} \alpha^{-1} \log \|\exp(\alpha a)\| \\ &= \sup_{\alpha>0} \alpha^{-1} \log e^{\alpha\gamma} \\ &= \gamma, \quad \text{proving the result.}\end{aligned}$$ //

Combining the previous lemma with Lemma 5.2 we obtain the following characterisation of hermitian elements.

5.5. **THEOREM.** *For a unital Banach Algebra* A, *the element* $a \in A$ *is hermitian if and only if* $[v(a+\lambda e)]^2 \leq p[(a+\lambda e)(a+\bar{\lambda}e)]$, *for all* $\lambda \in C$, *where* $p(b) = \rho(b)$, $v(b)$ *or* $\|b\|$ *for all* $b \in A$.

Proof. Sufficiency follows directly from Lemma 5.2. Hence it is only necessary to prove

$$[v(a+\lambda)]^2 \leq \rho[(a+\lambda)(a+\bar{\lambda})] \text{ for all } \lambda \in C$$

whenever a is hermitian. The other bounds then follow since $\rho(b) \leq v(b) \leq \|b\|$ for all $b \in A$. Hence, for hermitian $a \in A$ and $\lambda = \alpha+i\beta$ α,β real, we have $V(a+\alpha e)$ is real and since $V(a+\lambda e) = V(a+\alpha e)+i\beta$ we see that

$$\begin{aligned}[v(a+\lambda e)]^2 &= ([v(a+\alpha e)]^2+\beta^2) \\ &= [\rho(a+\alpha e)]^2 + \beta^2, \quad \text{by Lemma 5.4} \\ &= \rho[(a+\alpha e)^2] + \beta^2, \quad \text{by the spectral mapping theorem,}\end{aligned}$$

from which it also follows that $\sigma[(a+\alpha e)^2] \subseteq [0,\infty)$ and so

$\sigma[(a+\alpha e)^2]+\beta^2 = \sigma[(a+\alpha e)^2+\beta^2]$, since $\sigma(a+\alpha e)$ is real.

Therefore

$$[v(a+\lambda e)]^2 = \rho[(a+\alpha e)^2+\beta^2]$$
$$= \rho[(a+\lambda e)(a+\bar{\lambda}e)]. \qquad //$$

This theorem provides the motivation for our treatment of Banach*-algebras in the next chapter.

Using a generalisation to a theorem of S. Bernstein, A.M. Sinclair [1] has proved that $\rho(a+\alpha e) = \|a+\lambda e\|$, for all $\lambda \in C$, where a is an hermitian element of a unital Banach Algebra. This is a significantly stronger result than that of Lemma 5.4. The proof leads directly to the equation

$$\|a+\lambda\|^2 = ([\rho(a+\alpha e)]^2+\beta^2) \text{ where } \lambda = \alpha+i\beta \quad \alpha,\beta \text{ real.}$$

Combining this with the proof of Theorem 5.5 we would obtain the stronger characterisation that an element a of a unital Banach Algebra is hermitian if and only if

$$\|a+\lambda e\|^2 = \rho[(a+\lambda e)(a+\bar{\lambda}e)] \text{ for all } \lambda \in C.$$

A more elementary proof of Sinclair's result for the case when $\lambda = 0$ is given by Bonsall and Crabb [1], however the above characterisation does not appear to follow easily from it.

The last lemma in this section examines the set of hermitian elements in a Banach algebra. We will denote by $H(A)$ the set of all hermitian elements of the Banach Algebra A.

5.6. LEMMA. *For a unital Banach Algebra* A *the set of all hermitian elements of* A, $H(A)$, *is closed.*

Proof. Let a_n be a cauchy sequence of hermitian elements of A converging to $a \in A$. For any $\varepsilon > 0$ there exists N such that $\|a_n - a\| \leq \varepsilon$ for $n \geq N$. If $\lambda \in V(a)$ then $\lambda = f(a)$ for some $f \in D(e)$. Let $\lambda_n = f(a_n)$, then

$$|\lambda_n - \lambda| = |f(a_n - a)| \leq \|a_n - a\| \leq \varepsilon \quad \text{for } n \geq N.$$

So λ is the limit of the real sequence λ_n and therefore λ is real. //

For a unital Banach Algebra we have developed several characterisations of hermitian elements, the most important of which was Theorem 5.5, that is, a is hermitian if and only if $[v(a+\lambda e)]^2 \leq \rho[(a+\lambda e)(a+\bar{\lambda} e)]$ for all $\lambda \in \mathbb{C}$. This inequality may be compared to Corollary 4.3.1i), in which it was shown that $v(a+\lambda e) = \rho(a+\lambda e)$ implied $\overline{co}\,\sigma(a) = V(a)$. To establish this characterisation of hermitian elements we proved the fundamental result of I. Vidav, that for any hermitian element a $\overline{co}\,\sigma(a) = V(a)$. All of these results will be used in the development of later material, as will the last result that the set of hermitian elements is closed.

6. ELEMENTS WITH HERMITIAN DECOMPOSITION; NORMAL TYPE ELEMENTS; THE FUNCTIONAL CALCULUS FOR HERMITIAN ELEMENTS WHOSE POWERS ARE HERMITIAN.

In this section we establish properties of some special elements of a Banach Algebra. The motivation to study such elements comes from the theory of operators over a Hilbert space and the observation that every self-adjoint operator over a Hilbert space is hermitian in the sense of Definition 5.1.

6.1. **DEFINITION.** For a Banach Algebra A, we say the element $a \in A$ permits an *hermitian decomposition* if there exist hermitian elements $p,q \in A$ such that $a = p+iq$.

The content of the next lemma is also contained in Bonsall and Duncan [1].

6.2. **LEMMA.** *For a unital Banach Algebra* A *we have:*

i) *if* $a \in A$ *permits an hermitian decomposition then that decomposition is unique;*

ii) *the set of elements of* A *permitting an hermitian decomposition is a closed linear subspace of* A.

Proof. i) Assume $a = p+iq = p'+iq'$ where p,p',q,q' are hermitian, then $p-p' = i(q-q')$. Thus the hermitian element $p-p'$ equals i times the hermitian element $q-q'$ which is impossible unless $p = p'$ and $q = q'$.

ii) If $a,b \in A$ permit hermitian decompositions $a = p+iq$, $b = h+ik$, and if $\lambda = \alpha+i\beta$ α,β real, then

$$\begin{aligned}\lambda a+b &= (\alpha+i\beta)(p+iq) + (h+ik)\\ &= (\alpha p-\beta q+h) + i(\beta p+\alpha q+k)\end{aligned}$$

where $\alpha p-\beta q+h$ and $\beta p+\alpha q+k$ are hermitian, so $\lambda a+b$ permits an hermitian decomposition. Therefore the set of elements permitting hermitian decompositions is a linear subspace of A.

Now assume the sequence a_n converges to $a \in A$, where $a_n = p_n+iq_n$ p_n,q_n hermitian. For any $f \in D(e)$ and $\varepsilon > 0$ there exists N such that $|f(a_n-a_m)| \leqslant \|a_n-a_m\| \leqslant \varepsilon$ for $n,m \geqslant N$, but

$$\begin{aligned}|f(a_n-a_m)| &= |f(p_n-p_m+i(q_n-q_m))|\\ &= |f(p_n-p_m)+if(q_n-q_m)|\\ &= \sqrt{(|f(p_n-p_m)|^2+|f(q_n-q_m)|^2)}.\end{aligned}$$

Therefore $|f(p_n-p_m)|$, $|f(q_n-q_m)| \leqslant \varepsilon$ for all $f \in D(e)$ and so $v(p_n-p_m)$, $v(q_n-q_m) \leqslant \varepsilon$ and by Theorem 3.5

$$\|p_n-p_m\|,\ \|q_n-q_m\| \leqslant e\varepsilon.$$

Therefore by Lemma 5.6 there exist hermitian p,q such that p_n converges to p and q_n converges to q.

Now $\|a_n-(p+iq)\| \leq \|p_n-p\| + \|q_n-q\|$ so a_n converges to $p+iq$. Therefore $a = p+iq$ has an hermitian decomposition and so the set of elements with an hermitian decomposition is closed. //

A simple but important property of an element a which permits an hermitian decomposition $a = p+iq$ is that for $\alpha+i\beta \in V(a)$ α,β real, there exists an $f \in D(e)$ such that $f(p) = \alpha$ and $f(q) = \beta$. The next lemma makes use of this observation.

6.3. LEMMA. *For a unital Banach Algebra* A *and* $a \in A$ *permitting an hermitian decomposition there exists* $\lambda \in \mathbb{C}$, $|\lambda| = 1$, *such that* $v(a) = v(p)$ *where* $\lambda a = p+iq$ p,q *hermitian.*

Proof. By Corollary 2.5.1 $V(a)$ is compact, so there exists θ, $0 \leq \theta < 2\pi$, such that $v(a)e^{i\theta} \in V(a)$. Let $\lambda = e^{-i\theta}$ then $v(a) = v(\lambda a) \in V(\lambda a)$ so there exists $f \in D(e)$ such that $f(p) = v(a)$ where $\lambda a = p+iq$ p,q hermitian (possible by Lemma 6.2ii)). Hence $v(a) \leq v(p)$. But for any $f \in D(e)$

$$\begin{aligned} v(a) = v(\lambda a) &\geq |f(\lambda a)| \\ &= |f(p)+if(q)| \\ &= \sqrt{(f(p)^2+f(q)^2)} \end{aligned}$$

so $|f(p)| \leq v(a)$ and therefore

$$v(p) = v(a). \qquad //$$

We now turn our attention to an important subset of the elements which permit hermitian decompositions.

6.4. **DEFINITION.** For a Banach Algebra A we say the element $a \in A$ is a *normal type element* if a permits the hermitian decomposition $a = p+iq$ where $pq = qp$.

Such elements are also considered by Bonsall and Duncan [1] who refer to them as "normal" elements. A straightforward calculation shows that the set of normal type elements is closed under scalar multiplication and the addition of scalar multiples of the one, if present. That is, for a normal type element a, λa and $a+\lambda e$ are normal type elements for all $\lambda \in C$. Further if a has the hermitian decomposition $a = p+iq$ then $ap = pa$ and $aq = qa$.

An alternative proof of the next lemma is given by Bonsall and Duncan [1,5.14].

6.5. **LEMMA.** *For a unital Banach Algebra* A *and normal type element* $a \in A$, *we have* $\overline{\mathrm{co}}\, \sigma(a) = V(a)$.

Proof. Let b be any normal type element of A, then by Lemma 6.3 there exists $\lambda \in C$, $|\lambda| = 1$, such that $\lambda b = p+iq$, where p,q are hermitian and commute, with

$$v(b) = v(p)$$

$$= \rho(p) \quad \text{by Lemma 5.4.}$$

Now let $C_m(b,p)$ denote a maximal commutative subalgebra of A containing e,b and p. Then by the Gelfand theory there exists a multiplicative $f \in C_m(b,p)'$ such that $f(p) = \rho(p) = v(b)$. Again by the Gelfand theory $f(b) \in \sigma(b)$ so $|f(b)| \leqslant \rho(b)$. Therefore

$$v(b) = f(p) = \text{Re } f(\lambda b) \leqslant |f(b)| \leqslant \rho(b) \leqslant v(b)$$

or $v(b) = \rho(b)$.

In particular, then $v(a+\lambda e) = \rho(a+\lambda e)$ for all $\lambda \in C$ and so by Corollary 4.3.1i) $\overline{\text{co}}\ \sigma(a) = V(a)$. //

6.5.1. **COROLLARY.** *For a Banach Space* X *and normal type operator* $T \in B(X)$ *we have* $\overline{\text{co}}\ \sigma(T) = \overline{W(T)}$ *and so* $\overline{W(T)}$ *is convex.*

Proof. From Crabb's result [2] and Lemma 6.5 we have
$V(T) = \overline{\text{co}}\ \sigma(T) \subseteq \overline{W(T)} \subseteq V(T)$. //

We will make use of elements with hermitian decompositions, and the properties of normal type elements in the next chapter.

We now turn to developing a Functional Calculus for hermitian elements whose powers are also hermitian. That the powers of a hermitian element are not always hermitian is shown in a simple example by Crabb [1]. In fact as we will show in the next chapter, an hermitian element, all of whose powers are hermitian, may be identified with a self-adjoint operator over some Hilbert space. Thus the general setting of this functional calculus is only an apparent one. However as we see in the next chapter it is useful to have the tools of a functional calculus available to us in this setting.

Let A be a unital Banach Algebra and $h \in A$ be an hermitian element of A, all of whose powers are hermitian. Denote by $A(h)$ the closed subalgebra generated by e and h. It is readily seen that $A(h)$ is a commutative subalgebra of A all of whose elements permit an hermitian decomposition within $A(h)$ and hence A, and so are normal type elements. (Note, by Corollary 2.5.2 the concept of hermitian, and hence hermitian decomposition and normal type element, coincide for $A(h)$ and A.) Since every element a of $A(h)$ permits the hermitian decomposition $a = p+iq$, where by Lemma 6.2i) p and q are unique, we can define the mapping $a \mapsto a^* = p-iq$ of $A(h)$ into $A(h)$. A more detailed study of such a mapping is undertaken in the next chapter, here it is sufficient to have the notation available to us.

The functional calculus can now be developed in the standard way.

As h is hermitian $\sigma(h)$ is a closed, bounded and hence compact subset of the real line and so $C(\sigma(h))$, the set of all continuous functions $f : \sigma(h) \to \mathbb{C}$, is a complex Banach space under pointwise addition and the uniform norm;

$$\|f\| = \sup\{|f(\lambda)| : \lambda \in \sigma(h)\}.$$

Further for $f,g \in C(\sigma(h))$ the pointwise defined product $f \cdot g$ and conjugate function $\overline{f}$ are also in $C(\sigma(h))$.

Let $P(\sigma(h))$ be the subset of $C(\sigma(h))$ consisting of all polynomials p over $\sigma(h)$ defined by $p(\lambda) = \alpha_0+\alpha_1\lambda+\ldots+\alpha_n\lambda^n$ for $\lambda \in \sigma(h)$ and $\alpha_i \in C$ $i = 0,1,\ldots,n$, for any n. For such a $p \in P(\sigma(h))$ we can define $p(h) \in A(h)$ by $p(h) = \alpha_0 e+\alpha_1 h+\ldots+\alpha_n h^n$.

It is routine to check that such $p(h)$ satisfy the following:

i) $(p+p')(h) = p(h) + p'(h)$

ii) $(p \cdot p')(h) = p(h)p'(h)$

iii) $\overline{p}(h) = p(h)^*$.

Also by the spectral mapping theorem

iv) $\sigma(p(h)) = p(\sigma(h))$

and so

v) $v(p(h)) = \|p\|$.

This follows since $p(h)$ is a normal type element and so by Lemma 6.5

$$
\begin{aligned}
v(p(h)) &= \rho(p(h)) \\
&= \sup\{|\lambda| : \lambda \in \sigma(p(h))\} \\
&= \sup\{|p(\lambda)| : \lambda \in \sigma(h)\}, \text{ by iv)} \\
&= \|p\| .
\end{aligned}
$$

We can now state the basic theorem of the functional calculus.

6.6. THEOREM. *For a unital Banach Algebra* A *and hermitian element* $h \in A$ *whose powers are hermitian, there exist a unique map* $f \mapsto f(h) : C(\sigma(h)) \to A(h)$ *such that:*

i) $(f+g)(h) = f(h) + g(h)$

ii) $(f\cdot g)(h) = f(h)g(h)$

iii) $\overline{f}(h) = f(h)^*$

iv) $v(f(h)) = \|f\|$

v) *if* $f \in P(\sigma(h))$ *then* $f(h) = \alpha e + \alpha_1 h + \ldots + \alpha_n h^n$ *where* $f(\lambda) = \alpha_0 + \alpha_1\lambda + \ldots + \alpha_n\lambda^n$ *for* $\lambda \in \sigma(h)$.

Proof. From i) and v) of the previous page, the map $p \mapsto p(h)$ is a linear isometry from $P(\sigma(h))$ into $A(h)$ with the equivalent linear norm $v(\cdot)$.

By Weierstrass' approximation theorem $\overline{P(\sigma(h))} = C(\sigma(h))$ and so since $A(h)$ is complete $p \mapsto p(h)$ can be extended uniquely by continuity to a linear isometry $f \mapsto f(h)$ of $C(\sigma(h))$ into $A(h)$, which satisfies i), iv) and v) of the theorem. That ii) and iii) hold, now follows from the corresponding results for $p \mapsto p(h)$. //

Whereas for an arbitrary element of a Banach Algebra we have had to rely on establishing convergence of the corresponding infinite series to prove the existence of inverses, logs, exps, etc., the above theorem ensures us of their existence for an hermitian element h, whose powers are hermitian, provided the corresponding function is well defined on $\sigma(h)$.

6.7. LEMMA. *For a unital Banach Algebra* A *and hermitian element* $h \in A$ *whose powers are hermitian, we have:*

i) *if* $f \in C(\sigma(h))$ *is such that* $f(\sigma(h))$ *is real then* $f(h)$ *is hermitian*

ii) *if* $f \in C(\sigma(h))$ *is such that* $f(\sigma(h)) \subseteq [0,\infty)$ *then* $V(f(h)) \subseteq [0,\infty)$.

Proof. i) If $f \in C(\sigma(h))$ is real valued it may, by Weierstrass' approximation theorem, be approximated arbitrarily closely by real valued polynomials. If $p(\lambda) = \alpha_0+\alpha_1\lambda+\ldots+\alpha_n\lambda^n$ is real valued then α_i is real for $i = 0,1,\ldots,n$ and so $p(h) = \alpha_0 e+\alpha_1 h+\ldots+\alpha_n h^n$ is hermitian, therefore $f(h)$ may be approximated arbitrarily closely by hermitian elements and so by Lemma 5.6 $f(h)$ is hermitian.

ii) Since $f(\lambda) \geq 0$ for all $\lambda \in \sigma(h)$ we can define $g \in C(\sigma(h))$ by $g(\lambda) = \sqrt{f(\lambda)}$ for all $\lambda \in \sigma(h)$. Then since g is real valued, $g(h)$ is hermitian, by the proof of the first half of the lemma, and $f(h) = g(h)^2$, by Theorem 6.6ii). Therefore, by Lemma 5.4

$$\begin{aligned} V(f(h)) = V(g(h)^2) &= \overline{\text{co}}\ \sigma(g(h)^2) \\ &= \overline{\text{co}}\ \sigma(g(h))^2 \subseteq [0,\infty). \end{aligned}$$

//

We have now developed the basic theory of numerical ranges necessary for our purposes. In section 4 we saw how numerical ranges may be used to approximate the spectrum as closely as the basic properties of these sets will allow.
Hermitian elements were examined in section 5. The characterization of hermitian elements given by Theorem 5.5 is used in the next chapter to give an elementary proof that a unital Banach*-algebra, in which every self-adjoint element is hermitian, is a B*-algebra for some equivalent norm. Since the powers of a self-adjoint element are self-adjoint, we are able to apply the functional calculus of the last section to any self-adjoint element of a Banach*-algebra in which every self-adjoint element is hermitian. We use the results of section 6, on elements which permit an hermitian decomposition and normal type elements, to characterize these algebras among unital Banach algebras.